A PIECEWISE-LINEARIZED METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS: TWO-POINT BOUNDARY VALUE PROBLEMS

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SUMMARY

Piecewise-linearized methods for the solution of two-point boundary value problems in ordinary differential equations are presented. These problems are approximated by piecewise linear ones which have analytical solutions and reduced to finding the slope of the solution at the left boundary so that the boundary conditions at the right end of the interval are satisfied. This results in a rather complex system of non-linear algebraic equations which may be reduced to a single non-linear equation whose unknown is the slope of the solution at the left boundary of the interval and whose solution may be obtained by means of the Newton–Raphson method. This is equivalent to solving the boundary value problem as an initial value one using the piecewise-linearized technique and a shooting method. It is shown that for problems characterized by a linear operator a technique based on the superposition principle and the piecewise-linearized method may be employed. For these problems the accuracy of piecewise-linearized methods is of second order. It is also shown that for linear problems the accuracy of the piecewise-linearized method is superior to that of fourth-order-accurate techniques. For the linear singular perturbation problems considered in this paper the accuracy of global piecewise linearization is higher than that of finite difference and finite element methods. For non-linear problems the accuracy of piecewise-linearized methods is in most cases lower than that of fourth-order methods but comparable with that of second-order techniques owing to the linearization of the non-linear terms.

KEY WORDS   piecewise-linearized methods; two-point boundary value problems; singular perturbations; finite differences; finite elements

1. INTRODUCTION

Two-point boundary-value problems in ordinary differential equations occur in many branches of physics; examples include the two-dimensional, incompressible, boundary layer equations, one-dimensional heat transfer, etc. Numerical methods for the solution of two-point boundary value problems include finite difference, finite element and shooting methods.[1] The first approximate the derivatives that appear in the differential equation by a difference quotient and yield a system of algebraic equations whose unknowns are the values of the dependent variable at a finite number of points. Finite element methods look for a weak solution in a finite-dimensional functional space to the problem.[2] Finally, shooting methods solve a boundary value problem as an initial value one.

Flows at high Reynolds numbers or heat transfer phenomena at large Peclet numbers, i.e. convection-dominated flows, are also examples of two-point boundary value problems which are characterized by the presence of thin, viscous or heat-conducting layers where steep flow gradients occur, i.e. an inner region, and domains where the flow variables do not change much, i.e. an outer region. These flows are also examples of singular perturbation problems and have been a subject of intensive research.

Methods used to determine numerically the solution of two-point singular perturbation problems include finite difference, finite element, spectral and domain decomposition techniques. For example, the authors of Reference 3 employed spectral techniques. Domain decomposition methods that divide the interval into several regions, i.e. inner, outer, boundary or transition layers, have also been applied to singular perturbation problems.[4] The piecewise-linearized methods presented here are applied to some linear perturbation problems, provide analytical solutions, do not require the decomposition of the domain into inner and outer regions and yield more accurate results than those obtained with other techniques. Furthermore, since the piecewise-linearized methods presented here may use adaptive techniques based on, for example, the norm of the Jacobian matrix[5] which has to be determined in the linearization, they may use unequally spaced intervals and concentrate the grid points where steep gradients occur.

In this paper a technique based on the piecewise linearization of second-order ordinary differential equations which provides piecewise analytical solutions is presented. Continuity of the solution and its first-order derivative yields a rather complex system of non-linear algebraic equations for the values of the solution and its first-order derivative at each end point of each interval. This system may be reduced, after very long and cumbersome algebra, to a single non-linear algebraic equation whose unknown is the slope of the solution at the left boundary of the interval and this equation may be solved iteratively by means of the Newton–Raphson method. It can be shown that this is equivalent to solving a boundary value problem as an initial value one using the piecewise-linearized method and a shooting technique.

For linear, second-order, ordinary differential equations it is shown that the norm of the error depends on the norm of the difference between the exact and piecewise-linearized solutions of initial value problems which has been bounded in Reference 5.

The paper has been organized as follows. In Section 2 the piecewise-linearized method for the solution of two-point boundary value problems is described. In Section 3 the superposition principle and the piecewise-linearized method are combined to solve problems characterized by a linear operator and the order of the method is estimated for linear problems. The accuracy of piecewise-linearized methods is assessed by comparing their solutions with analytical ones when available and with those of second- and fourth-order-accurate shooting techniques for linear and non-linear ordinary differential equations in Section 4. Some linear singular perturbation problems which are of great relevance to convection-dominated flows are analysed in Section 5. Finally, Section 6 summarizes the main conclusions of the paper.

2. TWO-POINT BOUNDARY VALUE PROBLEMS

Consider the two-point boundary value problem

$$y'' = f(x, y, y'), \quad x \in (a, b), \tag{1}$$

$$y(a) = y_0, \qquad y(b) = y_f, \tag{2}$$

where the primes denote differentiation with respect to x.

If this problem has only one solution, Y, then this solution is also the same as that of the initial value problem whose differential equation is equation (1) and whose initial conditions are $y(a) = y_0$ and $y'(a) = Y'(a) = s$. This initial value problem can be solved using piecewise-linearized methods which

approximate the solution to the above boundary value problem in the following piecewise manner. The interval $[a, b]$ is divided into n subintervals $a = x_0 < x_1 < \cdots < x_n = b$. In each interval $I_{i+1} = [x_i, x_{i+1}]$, equation (1) is approximated by its linearization about x_i as

$$y''(x) = f_i + X_i(x - x_i) + G_i(y - y_i) + H_i(y' - y_i'), \quad x \in (x_i, x_{i+1}), \tag{3}$$

where $f_i = f(x_i, y_i, y_i')$, $X_i = (\partial f / \partial x)(x_i, y_i, y_i')$, $G_i = (\partial f / \partial y)(x_i, y_i, y_i')$ and $H_i = (\partial f / \partial y')(x_i, y_i, y_i')$ for $i = 0, 1, 2, \ldots, n - 1$.

Equation (3) may be solved analytically in I_{i+1} and its solution depends on two constants which are determined by imposing the continuity of y and y' at $i = 1, 2, \ldots, n - 1$, i.e.

$$y_{I_i}(x_i) = y_{I_{i+1}}(x_i) = y_i, \qquad y_{I_i}'(x_i) = y_{I_{i+1}}'(x_i) = y_i'. \tag{4}$$

At the left and right boundaries of the interval $[a, b]$,

$$y_{I_1}(x_0) = y_0, \qquad y_{I_1}'(x_0) = y_0', \qquad y_{I_n}(x_n) = y_f. \tag{5}$$

Equation (4) represents a system of $2n - 2$ non-linear algebraic equations with $2n - 2$ unknowns, i.e. y_i, $i = 1, 2, \ldots, n - 1$, and y_i', $i = 1, 2, \ldots, n - 1$, whose values depend on the solution of equation (3). Unfortunately, the solution of equations (3) and (4) is rather complex, because the solution in I_{i+1} depends on that in I_i which in turn depends on the solution in the interval I_{i-1}; however, it is in general possible to reduce equation (4) after rather long and cumbersome algebraic operations to only one non-linear algebraic equation with only one unknown, e.g. s, and the resulting equation may be solved by means of, for example, the Newton–Raphson method as follows. Let us denote the solution of the piecewise-linearized method with initial conditions $y(a) = y_0$ and $y'(a) = s$ by $y_s(x)$ and define the function $\mathbb{R} \to \mathbb{R}$ as $r(s) = y_s(b) - y_f$. The solution to equations (2) and (3) coincides with y_s when $r(s) = 0$. The slope at the left boundary of the interval, i.e. s, may be determined iteratively from the condition $r(s) = 0$ and this is equivalent to solving the boundary value problem of equations (2) and (3) by means of a simple shooting method using the piecewise-linearized technique for initial value problems.[5]

It must be noted that since X, G and H appear in equation (3) and these values are related to the variation in the second-order derivative with respect to x and the Jacobian, their values, i.e. norms, may be used to select the step size in such a manner that the number of intervals is increased where steep derivatives of the dependent variables occur. The adaption procedures developed in Reference 5 for initial value problems may also be used to solve boundary value ones.

It must also be noted that the piecewise-linearized methods presented here provide analytical, piecewise classical approximations to two-point boundary value problems, whereas finite element techniques provide weak solutions. Furthermore, since the homogeneous solution of equation (3) is of exponential type and may be approximated by the first two terms of its Taylor series expansion about x_i if $x_{i+1} - x_i$ is sufficiently small, the resulting linear approximation is similar to that used in linear finite element methods. However, the piecewise-linearized methods considered here also account for the dependence of f on x in an analytical manner. This is not to say that piecewise-linearized methods are in general better than finite element techniques, since the differentiation with respect to the dependent and independent variables in the former is a noisy operation, whereas the weak formulation employed in finite element techniques is a smoothing operation owing to the quadratures used in these methods.

3. LINEAR OPERATORS

When f is of the form $f(x, y, y') = A(x) + B(x)y + C(x)y'$, it is not necessary to use the shooting method to solve the boundary value problem, because if y_1 and y_2 are solutions of equation (1) and c_1 and c_2 are two constants satisfying $c_1 + c_2 = 1$, then $c_1 y_1 + c_2 y_2$ is also a solution of equation (1). In order to determine the values of c_1 and c_2, one must solve the systems of linear equations

$$y_1'' = A(x) + B(x)y_1 + C(x)y_1', \qquad y_2'' = A(x) + B(x)y_2 + C(x)y_2',$$
$$y_1(a) = y_0, \qquad\qquad\qquad y_2(a) = y_0,$$
$$y_1'(a) = y_{01}', \qquad\qquad\qquad y_2'(a) = y_{02}'. \qquad\qquad (6)$$

where $y_{01}' \neq y_{02}'$, and c_1 and c_2 are the solution to the system of linear algebraic equations

$$c_1 + c_2 = 1, \qquad c_1 y_1(b) + c_2 y_2(b) = y_f.$$

Note that when $A(x)$ is a linear function of x, and B and C are constants, the piecewise-linearized method gives the exact solution to the boundary value problem, provided that there are no round-off errors.

3.1. Error estimation

Let y be the solution to equations (1) and (2) when f is linear, i.e. $y = c_1 y_1 + c_2 y_2$, and $\bar{y}_1$ and $\bar{y}_2$ their approximations by the piecewise-linearized method. Let $\bar{c}_1$ and $\bar{c}_2$ be the solution to the linear system

$$\bar{c}_1 + \bar{c}_2 = 1, \qquad \bar{c}_1 \bar{y}_1(b) + \bar{c}_2 \bar{y}_2(b) = y_f.$$

Note that since

$$c_i = \frac{y_f - y_i(b)}{y_1(b) - y_2(b)} \quad \text{and} \quad \bar{c}_i = \frac{y_f - \bar{y}_i(b)}{\bar{y}_1(b) - \bar{y}_2(b)}, \quad i = 1, 2, \qquad (7)$$

the convergence in the infinite, i.e. maximum, norm of $\bar{y}_i$ to y_i guarantees the convergence of $\bar{c}_i$ to c_i. Furthermore, since

$$\|y - (\bar{c}_1 \bar{y}_1 + \bar{c}_2 \bar{y}_2)\| \leqslant |c_1| \|y_1 - \bar{y}_1\| + |c_1 - \bar{c}_1| \|\bar{y}_1\| + |c_2| \|y_2 - \bar{y}_2\| + |c_2 - \bar{c}_2| \|\bar{y}_2\|, \qquad (8)$$

where $\|c_i - \bar{c}_i\|$ may be estimated from equation (7), y_i and $\bar{y}_i$ are the solutions of initial value problems and $\|y_i - \bar{y}_i\|$, $i = 1, 2$, can be bounded as for initial value problems.[5] Such a bound indicates that the error in the approximation to the boundary value problem introduced by the piecewise-linearized method is $O(\Delta x^2)$, where Δx denotes the largest step size.

4. PRESENTATION OF RESULTS

The accuracy of the piecewise-linearized method presented in this paper has been assessed by comparing its solutions with the analytical ones when available and with those obtained with the explicit Runge–Kutta methods of orders two and four, which are referred to as SHE and SHR respectively. The piecewise-linearized method for the second-order ordinary differential equations (see equation (3)) which uses shooting is here referred to as SHL.

For linear operators the superposition principle is applicable and the piecewise-linearized method presented here is referred to as LM1. Since equation (3) may be written as a system of two first-order ordinary differential equations, the piecewise-linearized methods presented in Reference 5 for first-order ordinary differential equations may be applied; these methods are here referred to as LM2.

All the methods considered in this paper have been used to determine the solutions to the following problems taken from References 3, 4 and 6.

P1:[4] $\varepsilon y'' + y' - y = 0$, $y(0) = y(1) = 1$.

Exact solution: $y(x) = \dfrac{(e^{m_1} - 1)e^{m_2 x} + (1 - e^{m_2})e^{m_1 x}}{e^{m_1} - e^{m_2}}$, $m_{1,2} = \dfrac{-1 \bullet \sqrt{(1 + 4\varepsilon)}}{2\varepsilon}$.

P2:[4] $\varepsilon y'' + y' = 1 + 2x$, $y(0) = 0$, $y(1) = 1$.

Exact solution: $y(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)\dfrac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}$.

P3:[3] $-y'' + 400y = -400\cos^2(\pi x) - 2\pi^2 \cos(2\pi x)$, $y(0) = y(1) = 0$.

Exact solution: $y(x) = \dfrac{e^{-20}}{1 + e^{-20}} e^{20x} + \dfrac{1}{1 + e^{-20}} e^{-20x} - \cos^2(\pi x)$.

P4:[3] $\varepsilon y'' - y = 0$, $y(-1) = 1$, $y(1) = 2$.

Exact solution: $y(x) = \dfrac{2e^{1/\sqrt{\varepsilon}} - e^{-1/\sqrt{\varepsilon}}}{e^{2/\sqrt{\varepsilon}} - e^{-2/\sqrt{\varepsilon}}} e^{x/\sqrt{\varepsilon}} + \dfrac{e^{1/\sqrt{\varepsilon}} - 2e^{-1/\sqrt{\varepsilon}}}{e^{2/\sqrt{\varepsilon}} - e^{-2/\sqrt{\varepsilon}}} e^{-x/\sqrt{\varepsilon}}$.

P5:[3] $y'' + 5y' + 10000y = -500\cos(100x)e^{-5x}$, $y(0) = 0$, $y(1) = \sin(100)e^{-5}$.

Exact solution: $y(x) = \sin(100x)e^{-5x}$.

P6:[6] $y'' = 1 + (y')^2$, $y(0) = 0$, $y(\pi/4) = (\ln 2)/2$.

Exact solution: $y(x) = -\ln[\cos(x)]$.

P7:[6] $y'' = \sqrt{[1 + (y')^2]}$, $y(0) = 1$, $y(1) = \cosh(1)$.

Exact solution: $y(x) = \cosh(x)$.

P8:[6] $3y''y' = 2y$, $y(0) = 1$, $y(3) = 8$.

Exact solution: $y(x) = \left(\dfrac{x}{3} + 1\right)^3$.

In addition to the above eight test problems, numerical solutions were obtained for other non-linear problems. However, the results obtained exhibited similar trends to those reported in this section and are not presented here.

Since the differential operator in P1–P5 is linear, the superposition principle described in Section 3 may be used to determine their solutions. Problems P1, P2 and P4 exhibit boundary layers at the boundaries of the interval and are representative of one-dimensional, convection-dominated flows if $\varepsilon \ll 1$, whereas P6–P8 are non-linear. Problems P3 and P6–P8 may be considered as examples of one-dimensional heat transfer phenomena with internal heat generation.

In simple shooting methods the initial value of s was set to $s_0 = (y_f - y_0)/(b - a)$ and the initial value problems were solved with $s = s_i = s_{i-1} + \Delta s$ until $|r(s)| \leqslant \tau$, where $\tau = 5 \times 10^{-7}$, $\Delta s = -r'(s)/r(s)$, $r'(z)$ is approximated by $[r(z + \Delta z) - r(z)]/\Delta z$ and $\Delta z = 1 \times 10^{-8}$.

The results obtained with the methods described in this paper are presented in Figures 1–4 and Tables I and II. In problems P1, P2 and P4, $\varepsilon = 10^{-2}$ and the piecewise-linearized method for solving initial value problems was employed as a global technique, i.e. only one interval was considered in order to avoid numerical errors. However, for problems P3 and P5, 1000 subintervals were considered to approximate accurately the non-homogeneous term in the differential equation.

Figures 1–4 illustrate the exact solution and the local errors introduced by the numerical methods considered in this paper. Figure 1 (top) clearly shows a thin boundary layer at the left of the interval and indicates that in the boundary layer the errors of SHL are smaller than those of SHE and SHR. In the outer region, i.e. for x greater than about 0·1, SHL is still more accurate than SHE but slightly less so
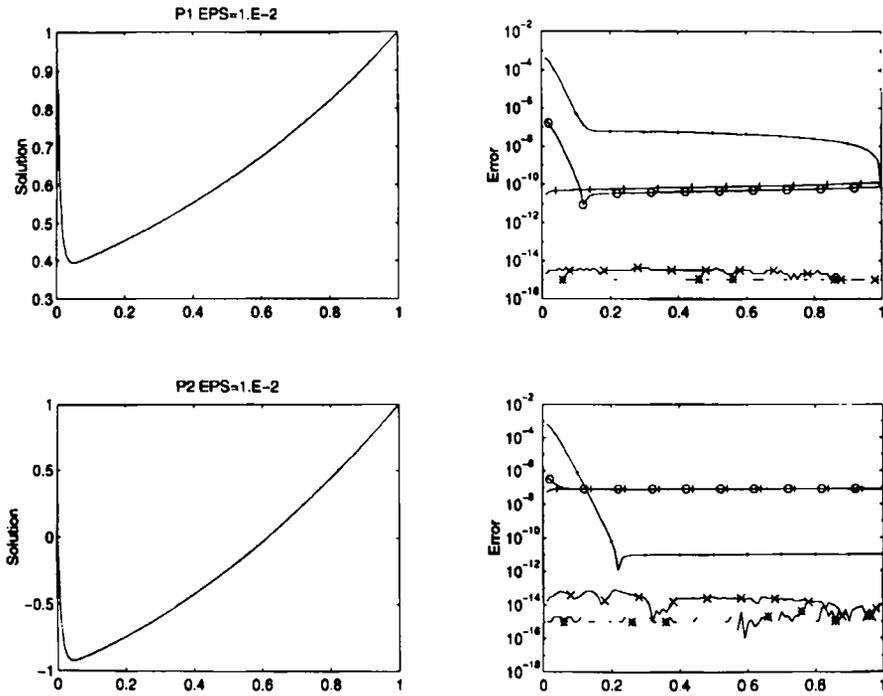
Figure 1. Exact solution (left) and local errors (right) for problems P1 (top) and P2 (bottom) ($\varepsilon = 10^{-2}$; SHE = ●; SHR = O; SHL = +; LM1 = *; LM2 = ×)

than SHR. The accuracy of LM1 and LM2 is higher than that of the other methods considered in this paper, because P1 is a linear problem and does not have non-homogeneous terms. Note that LM1 and LM2 yielded the exact solution at some points; as a consequence, the local errors of these two methods cannot be represented as continuous lines in semilogarithmic plots.

P2 also exhibits a boundary layer at the left of the interval as indicated in Figure 1 (bottom); however, this problem has a non-homogeneous term. Figure 1 (bottom) exhibits similar trends to those of Figure 1 (top) in the boundary layer; in the outer region the accuracy of SHL is about the same as that of SHR, which in turn is less accurate than that of SHE. The accuracy of the latter is lower than that of LM1 and LM2.

For P3 the results presented in Figure 2 (top) indicate that the accuracy of SHL is about the same as that of LM1 and LM2 and higher and lower respectively than that of SHE and SHR respectively. The lower performance of piecewise- linearized methods for P3 may be attributed to the non-homogeneous terms in the differential equation and their linearization with respect to x.

The results presented in Figure 2 (bottom) for P4 exhibit a boundary layer at each end of the interval, while the outer solution is zero. This figure indicates that for P4 the piecewise-linearized methods presented in this paper are more accurate than SHE and SHR; only for $x \geqslant 0$ is the accuracy of SHL comparable with that of SHR. Figure 2 (bottom) also indicates that LM1 is more accurate than the other methods considered in this paper.

The solution of P5 is a damped oscillation as shown in Figure 3 (top), which indicates that the least accurate method considered in this paper is SHE. This figure also indicates that the local errors decrease on average as x increases and that the local errors of SHR, SHL, LM1 and LM2 seem to tend to the same value at the right boundary of the interval.
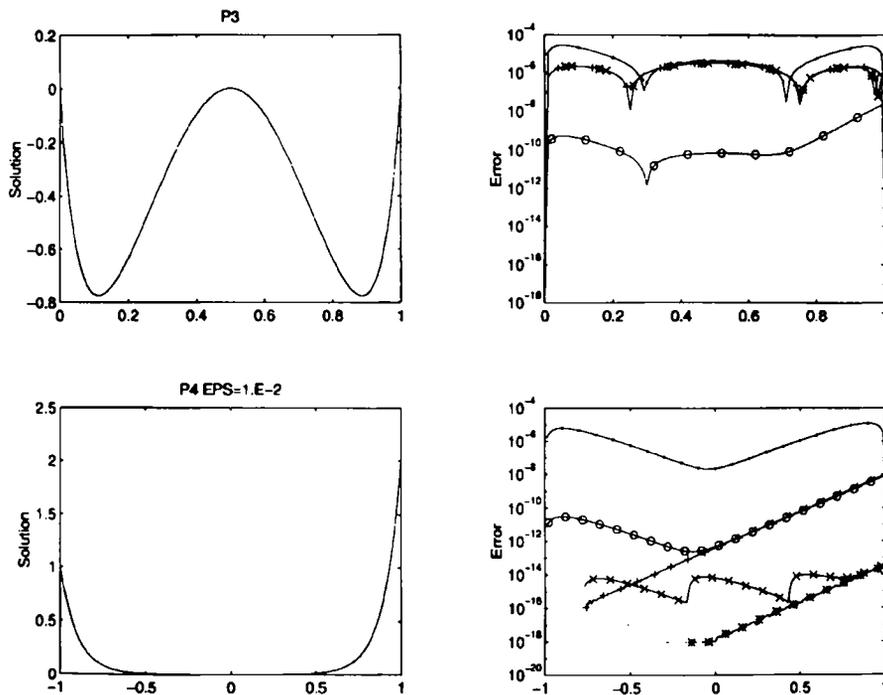
Figure 2. Exact solution (left) and local errors (right) for problems P3 (top) and P4 (bottom, $\varepsilon = 10^{-2}$) (SHE = ●; SHR = ○; SHL = +; LM1 = *; LM2 = ×)
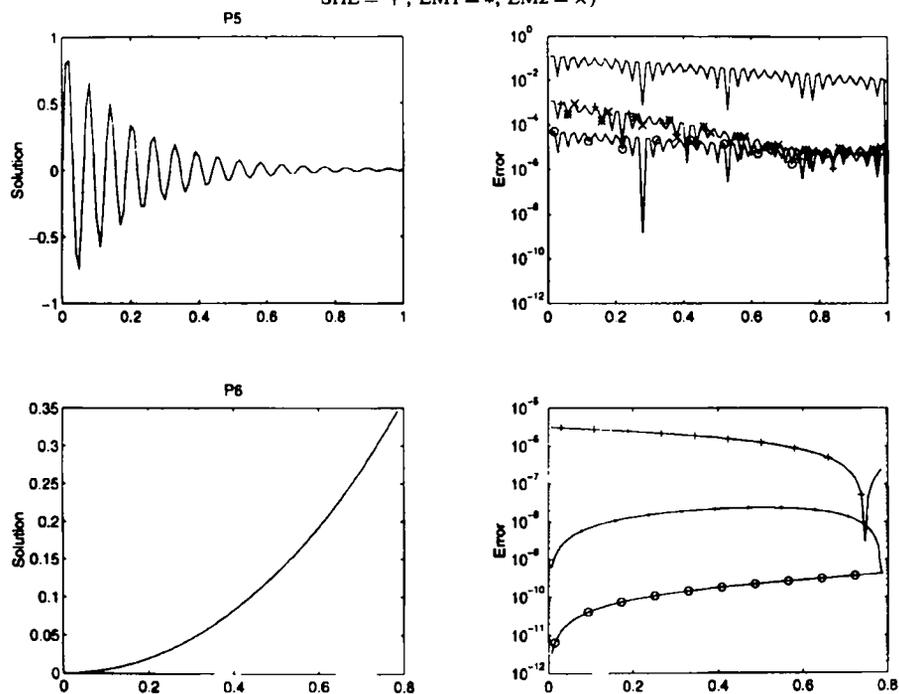
Figure 3. Exact solution (left) and local errors (right) for problems P5 (top) and P6 (bottom) (SHE = ●; SHR = ○; SHL = +; LM1 = *; LM2 = ×)
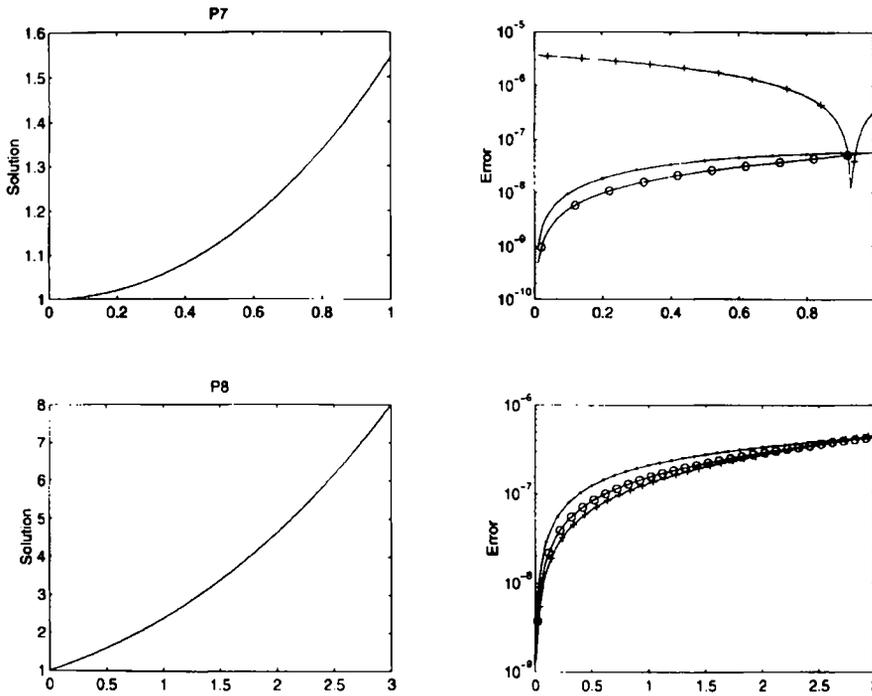
Figure 4. Exact solution (left) and local errors (right) for problems P7 (top) and P8 (bottom) (SHE = ●; SHR = O; SHL = +)

Figures 3 (bottom) and 4 correspond to P6 and P7–P8 respectively and have exact solutions which are monotonically increasing functions of x. For P6 and P7 the results presented in Figures 3 (bottom) and 4 (top) respectively indicate that SHR is more accurate than SHE, which in turn is more accurate than SHL. The relatively poorer accuracy of SHL in P6 and P7 is due to the non-linearities of these problems. However, the trends of the local errors exhibited in these two figures are not valid for all non-linear problems, as indicated in Figure 4 (bottom). This figure indicates that for P8 the accuracy of SHL is higher than that of SHR; the accuracy of the latter is in turn higher than that of SHE.

The arithmetic mean and maximum value of the absolute errors introduced by the methods analysed in this paper are given in Tables I and II for P1–P8. Table I shows that for P1, P2 and P4 the piecewise-linearized methods are more accurate than the other techniques considered in this paper and their errors are due to the finite arithmetic of the computer. The piecewise-linearized method with shooting, i.e. SHL, is more accurate than SHE and SHR for the same error tolerance in the shooting, i.e. for the same τ; its accuracy is, however, lower than that of LM1 and LM2.

For P3 and P5, which are linear problems but where $A(x)$ is a non-linear function of x, SHR is more accurate than piecewise- linearized techniques; the latter are in turn more accurate than SHE. The reason for the large differences between the accuracy of SHR and that of the piecewise-linearized techniques is due to the non-linear dependence of the right-hand side of the differential equation on x, i.e. the term X in equation (3).

Table II shows that for problems P6–P8, which were solved with a constant step size equal to 10^{-3}, the piecewise-linearized techniques are less accurate than the second- and fourth-order Runge–Kutta methods, except for problem P8 for which SHL yields smaller errors. Note that P6–P8 are non-linear problems; therefore the superposition principle is not applicable and LM1 and LM2 cannot be used.

Table I. Errors in problems P1–P5

	E_{mean}				
	P1	P2	P3	P4	P5
SHR	5.3798244×10^{-9}	8.7923933×10^{-8}	1.5717124×10^{-9}	$4.0737186 \times 10^{-10}$	1.3951807×10^{-5}
SHE	1.0448501×10^{-5}	1.6081193×10^{-5}	8.5375035×10^{-6}	2.4767466×10^{-6}	3.4900979×10^{-2}
SHL	$7.6979587 \times 10^{-11}$	7.9831789×10^{-8}	1.7653451×10^{-6}	$4.6631595 \times 10^{-10}$	1.5390086×10^{-4}
LM1	$4.0561613 \times 10^{-16}$	$1.5661461 \times 10^{-15}$	1.7174288×10^{-6}	$1.7341918 \times 10^{-15}$	1.5390046×10^{-4}
LM2	$2.4936049 \times 10^{-15}$	$2.3605464 \times 10^{-14}$	1.7458321×10^{-6}	$4.6623702 \times 10^{-15}$	1.5390047×10^{-4}

	E_{max}				
	P1	P2	P3	P4	P5
SHR	2.1801232×10^{-7}	3.7784252×10^{-7}	2.8382341×10^{-8}	7.5550197×10^{-9}	5.0631701×10^{-5}
SHE	4.2439685×10^{-4}	6.4831278×10^{-4}	2.7438932×10^{-5}	1.2148157×10^{-5}	1.2618765×10^{-1}
SHL	$1.2229695 \times 10^{-10}$	8.1102103×10^{-8}	3.2901473×10^{-6}	8.9195402×10^{-9}	1.1597276×10^{-3}
LM1	$9.9920072 \times 10^{-16}$	$7.9936058 \times 10^{-15}$	3.2902051×10^{-6}	$3.0198066 \times 10^{-14}$	1.1597258×10^{-3}
LM2	$4.0523140 \times 10^{-15}$	$8.2933660 \times 10^{-14}$	3.2901688×10^{-6}	$3.0198066 \times 10^{-14}$	1.1597259×10^{-3}

The results presented in Figures 1–4 and Tables I and II indicate that the piecewise-linearized methods presented in this paper have an accuracy of $O(\Delta x^2)$, where Δx denotes the largest step size, in agreement with both the error bounds derived in Reference 5 for initial value problems and the fact that these methods are based on a first-degree polynomial approximation to the right-hand side of equation (1) (see equation (3)).

5. PERTURBATION PROBLEMS

Problems P1, P2 and P4 are linear perturbation problems which have also been studied with $\varepsilon = 10^{-3}$ and 10^{-4}. Note that as the value of ε is decreased, the boundary layer thickness decreases and the gradients become steeper in the boundary layers.

Although a global approximation is advisable for linear problems to minimize round-off errors, the convergence of the series used in LM2[5] requires the use of a local approximation for these problems. In this paper a step size equal to 10^{-4} was used. Some sample results are presented in Figures 5–7. Figure 5

Table II. Errors in problems P6–P8

	E_{mean}		
	P6	P7	P8
SHR	$1.8556899 \times 10^{-10}$	2.6262750×10^{-8}	2.2109608×10^{-7}
SHE	1.5602134×10^{-8}	3.6351361×10^{-8}	2.6017789×10^{-7}
SHL	1.6170821×10^{-6}	1.6130822×10^{-6}	1.9240723×10^{-7}

	E_{max}		
	P6	P7	P8
SHR	$4.1995701 \times 10^{-10}$	5.6784810×10^{-8}	4.3750403×10^{-7}
SHE	2.3960043×10^{-8}	5.6736410×10^{-8}	4.3710214×10^{-7}
SHL	3.1260014×10^{-6}	3.2785625×10^{-6}	4.1962271×10^{-7}

indicates that the largest errors occur in the boundary layer; in the outer region the errors are nearly constant despite the fact that the solution increases monotonically in that region. Furthermore, the local errors in the outer region are nearly independent of the thickness of the boundary layer.

Figure 5 also shows that the accuracy of LM1 is comparable with that of LM2 and higher than that of SHL; the accuracy of the latter is higher than that of SHR, which in turn is more accurate than SHE.

For P2 the results illustrated in Figure 6 indicate that the local errors depend slightly on the value of ε. Note that the same step size was used for $\varepsilon = 10^{-3}$ and 10^{-4} despite the fact that the boundary layer thickness decreases as ε is decreased. For $\varepsilon = 10^{-3}$, Figure 6 (top) indicates that the local errors of SHL, SHR and SHE are nearly identical but higher than those of LM1 and LM2. For $\varepsilon = 10^{-4}$ (see Figure 6 (bottom)), however, SHL is slightly more accurate than SHE, which in turn is slightly more accurate than SHR.

For problem P4 the shooting methods considered in this paper do not converge, because the determination of the slope s is an ill-conditioned problem; therefore finite difference and finite element methods were used to solve this problem and their results were compared with those of LM1 and LM2. The finite difference method employed here is referred to as FD, uses central differences to discretize the second-order derivative and may be written as

$$\frac{\varepsilon}{h_i^2} y_{i-1} - \left(\varepsilon \frac{h_i + h_{i+1}}{h_i^2 h_{i+1}} + 1 \right) y_i + \frac{\varepsilon}{h_i h_{i+1}} y_{i+1} = 0, \quad i = 1, 2, \ldots, n-1,$$

$$y_0 = 1, \qquad y_n = 2,$$

(9)

where $h_i = x_i - x_{i-1}$.
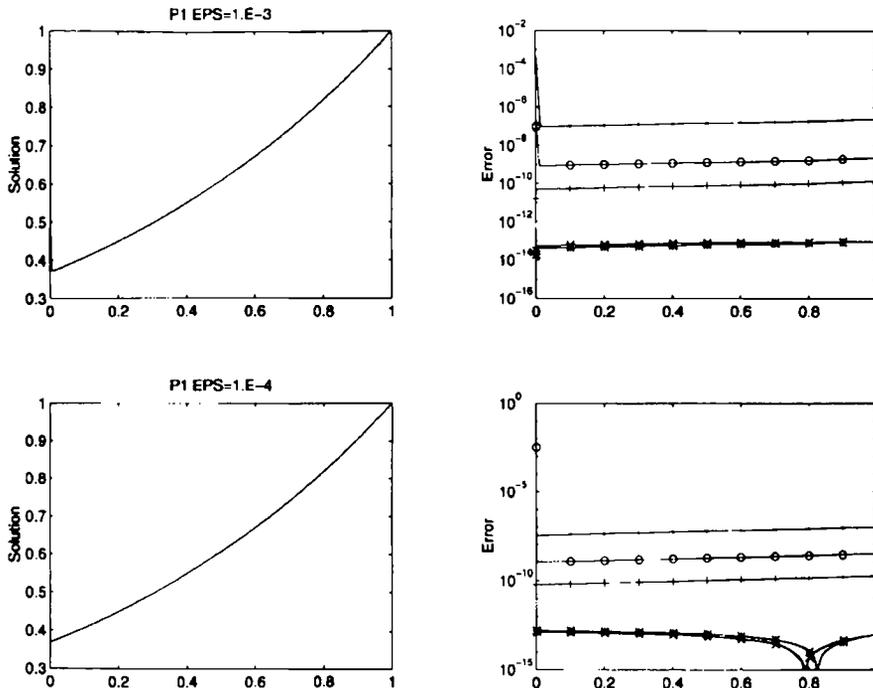
Figure 5. Exact solution (left) and local errors (right) for problem P1 (top, $\varepsilon = 10^{-3}$; bottom, $\varepsilon = 10^{-4}$; SHE = ●; SHR = ○; SHL = +; LM1 = *; LM2 = ×)
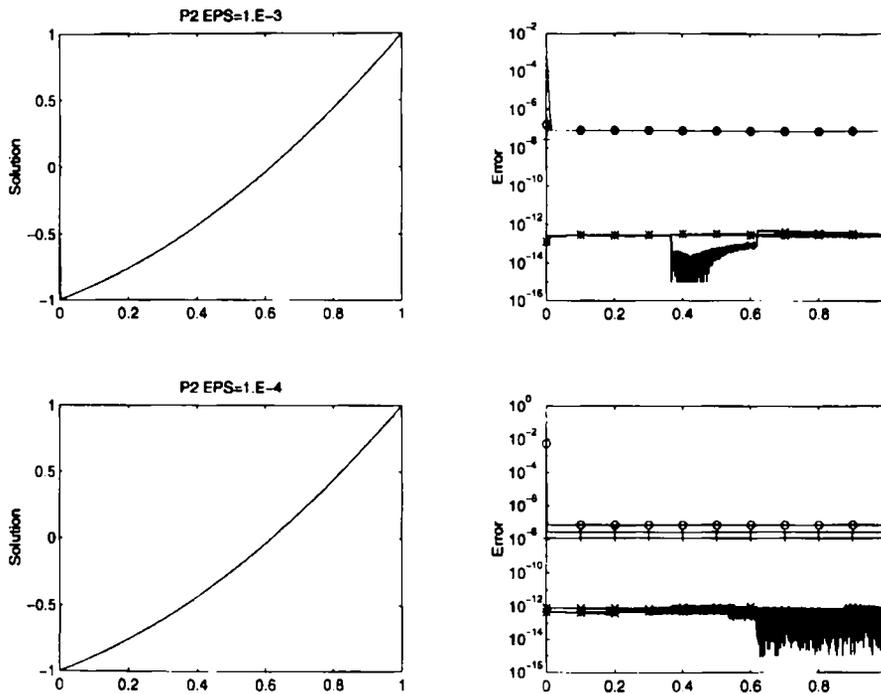
Figure 6. Exact solution (left) and local errors (right) for problem P2 (top, $\varepsilon = 10^{-3}$; bottom, $\varepsilon = 10^{-4}$; SHE = ●; SHR = O; SHL = +; LM1 = *; LM2 = ×)

Equation (9) is a tridiagonal linear system of algebraic equations which may be solved by means of the algorithm described in Reference 7 for this kind of system. Here a fixed step size $h = 10^{-4}$ was employed.

The finite element method used here is denoted by FE and based on C^0-linear Lagrange elements, i.e. the boundary value problem is approximated in the finite-dimensional space of the functions which are linear in every subinterval $[x_i, x_{i+1}]$, where $x_i = a + ih$, using a constant step size $h = 10^{-4}$. In this way one obtains a tridiagonal linear system which is solved using the same technique as in the finite difference method.

The finite difference and finite element results for P4 are compared with those of LM1 in Figure 7 for two different values of ε. This figure indicates that the finite difference method yields nearly the same results as those of the finite element technique. The local errors of these two methods decrease from the left boundary to the middle of the interval and then increase as the right boundary of the interval is approached, i.e. the errors increase as the magnitude of the gradients increases. Similar trends are exhibited by LM1, whose errors nearly coincide with those of the finite difference method for $x \geqslant 0$ and $\varepsilon = 10^{-3}$. For $\varepsilon = 10^{-4}$ the local errors of LM1 are lower than those of the finite element technique. The results presented in Figure 7 are perhaps a little bit counter-intuitive, since for the same step size one would expect that the errors for $\varepsilon = 10^{-4}$ would be larger than those for $\varepsilon = 10^{-3}$ on account of the boundary layer thickness and the resolution of the steep gradients.

Table III gives the arithmetic mean and maximum of the absolute errors for P1 and P2 and indicates that LM1 and LM2 are more accurate than the shooting techniques SHE, SHR and SHL. The last is the most accurate of the shooting methods employed in this paper.
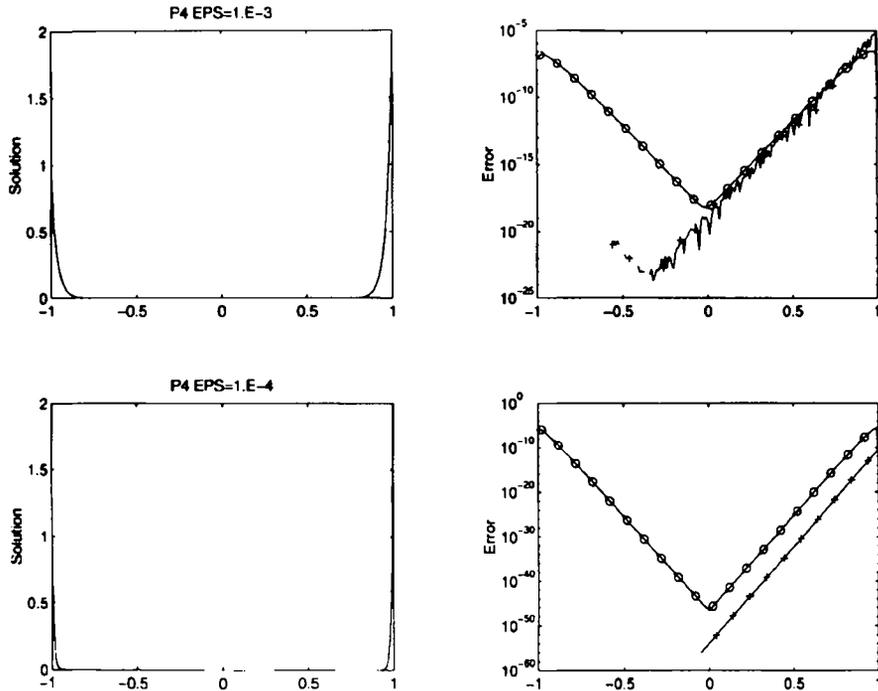
Figure 7. Exact solution (left) and local errors (right) for problem P4 (top, $\varepsilon = 10^{-3}$; bottom, $\varepsilon = 10^{-4}$; FD = ●; FE = ○; LM1 = +)

Table III. Perturbation problems P1 and P2

	E_{mean}			
	P1 ($\varepsilon = 1 \times 10^{-3}$)	P1 ($\varepsilon = 1 \times 10^{-4}$)	P2 ($\varepsilon = 1 \times 10^{-3}$)	P2 ($\varepsilon = 1 \times 10^{-4}$)
SHR	1.3963621×10^{-9}	1.8296470×10^{-9}	8.1761655×10^{-8}	7.0219510×10^{-8}
SHE	1.6418604×10^{-7}	6.1017710×10^{-8}	8.9891030×10^{-8}	2.7353256×10^{-8}
SHL	$8.3041258 \times 10^{-11}$	$1.0490587 \times 10^{-10}$	8.1757715×10^{-8}	1.1717284×10^{-8}
LM1	$6.3273919 \times 10^{-14}$	$9.6295909 \times 10^{-14}$	$2.9313697 \times 10^{-13}$	$3.9241821 \times 10^{-13}$
LM2	$7.5303350 \times 10^{-14}$	$7.6996715 \times 10^{-14}$	$2.6244947 \times 10^{-13}$	$6.6607115 \times 10^{-13}$

	E_{max}			
	P1 ($\varepsilon = 1 \times 10^{-3}$)	P1 ($\varepsilon = 1 \times 10^{-4}$)	P2 ($\varepsilon = 1 \times 10^{-3}$)	P2 ($\varepsilon = 1 \times 10^{-4}$)
SHR	2.2150440×10^{-9}	2.9089741×10^{-9}	8.2981812×10^{-8}	7.0921783×10^{-8}
SHE	6.1076998×10^{-7}	9.6316134×10^{-8}	9.0397327×10^{-7}	2.7626911×10^{-8}
SHL	$1.3197399 \times 10^{-10}$	$1.6698298 \times 10^{-10}$	8.2575331×10^{-8}	1.1834457×10^{-8}
LM1	$9.4035890 \times 10^{-14}$	$1.6203705 \times 10^{-13}$	$3.4197645 \times 10^{-13}$	$9.6600505 \times 10^{-13}$
LM2	$9.4035890 \times 10^{-14}$	$1.3294921 \times 10^{-13}$	$5.2050031 \times 10^{-13}$	$1.1129986 \times 10^{-12}$

Table IV. Perturbation problem P4

	E_{mean}		E_{max}	
	P4 ($\varepsilon = 1 \times 10^{-3}$)	P4 ($\varepsilon = 1 \times 10^{-4}$)	P4 ($\varepsilon = 1 \times 10^{-3}$)	P4 ($\varepsilon = 1 \times 10^{-4}$)
LM1	$2 \cdot 1781684 \times 10^{-13}$	$2 \cdot 9529322 \times 10^{-13}$	$1 \cdot 1870060 \times 10^{-11}$	$3 \cdot 7519987 \times 10^{-11}$
LM2	$1 \cdot 1794762 \times 10^{-5}$	$3 \cdot 1120150 \times 10^{+53}$	$7 \cdot 7030069 \times 10^{-4}$	$3 \cdot 6779893 \times 10^{+55}$
FE	$1 \cdot 9502861 \times 10^{-8}$	$5 \cdot 7255727 \times 10^{-8}$	$3 \cdot 0614774 \times 10^{-7}$	$3 \cdot 0656474 \times 10^{-6}$
FD	$1 \cdot 9502936 \times 10^{-8}$	$5 \cdot 7256288 \times 10^{-8}$	$3 \cdot 0614912 \times 10^{-7}$	$3 \cdot 0656766 \times 10^{-6}$

The mean and maximum errors for problem P4 given in Table IV indicate that the global technique LM1 is more accurate than FD and FE when a fixed step size $h = 10^{-4}$ is used in the latter. Table IV also indicates that LM2 fails to converge owing to numerical problems for $\varepsilon = 10^{-4}$.

As stated previously, P1–P8 were solved by means of the piecewise-linearized methods presented in this paper with constant step sizes. These techniques can also employ variable step sizes depending on the norm of the Jacobian matrix as indicated in Reference 5 for initial value problems. In fact, since these methods require the evaluation of the Jacobian at each end of the subintervals into which the domain is divided, this evaluation may be used to adapt the step size in a rather simple manner. Furthermore, since piecewise-linearized methods for second-order ordinary differential equations account for the derivatives of the right-hand side terms with respect to the independent variables, these derivatives may also be employed to adapt the step size when the Jacobian matrix does not change much from a subinterval to the next. It is to be expected that piecewise-linearized methods which employ variable step sizes will be more economical than those which use a fixed step, especially in singular perturbation problems characterized by steep boundary, shock, transition or internal layers.

When the Jacobian matrix is evaluated at each end of each subinterval, the piecewise-linearized methods employed in this paper may not be competitive in terms of computational cost and efficiency with other techniques; however, they do provide piecewise analytical solutions, whereas other techniques such as finite difference methods employ interpolants to obtain continuous solutions.

Although in the piecewise-linearized methods employed in this paper both the Jacobian matrix and the derivatives of the right-hand side terms with respect to the independent variables were evaluated at each left end of each subinterval, this may not be required when both this matrix and these derivatives do not vary much between successive intervals. It must also be noted that when f in equation (1) is not a function of y and y', this equation may in general be integrated analytically, whereas the solution of equation (3) is a parabola in each interval. This result is not at all surprising, since the piecewise-linearized methods employed in this paper are based on first-degree polynomial approximations to the right-hand side of equation (1).

6. CONCLUSIONS

Piecewise-linearized methods based on the linearization of ordinary differential equations have been used to solve two-point boundary value problems of second-order differential equations. These methods provide piecewise analytical solutions and are reduced to finding the slope of the solution at the left boundary of the interval which is consistent with the boundary conditions at the right end of the interval. This results in a rather complex system of non-linear algebraic equations owing to the requirement that the solution and its first-order derivative be continuous at the boundaries of the subintervals into which the domain has been divided. However, it is possible to reduce this system to a single non-linear

equation whose unknown is the slope of the solution at the left boundary; the roots of this equation may be found by means of the Newton–Raphson method. This is equivalent to solving the boundary value problem as an initial value one using the piecewise-linearized technique with a shooting method.

For problems characterized by a linear operator a method based on the superposition principle and the piecewise-linearized technique whose approximation error is of the same order as that for initial value problems has been developed.

For linear problems, i.e. linear operators with constant coefficients and non-homogeneous terms which are linear functions of the independent variable, a global approximation of the equation is advisable to avoid round-off errors in piecewise-linearized techniques. For non-linear ones it is necessary to divide the interval into subintervals. Although in this paper the step size has been kept constant, the step size in piecewise-linearized techniques can be controlled and adapted depending on the norm of the Jacobian matrix.

The accuracy of piecewise-linearized methods for boundary value problems of second-order ordinary differential equations has been assessed by comparing their solutions with analytical ones when available and with those of other shooting techniques. It has been shown that piecewise-linearized methods are more accurate than other techniques for linear problems and comparable with a second-order-accurate Runge–Kutta method with shooting. For linear singular perturbation problems it has been shown that global piecewise-linearized methods are more accurate than second-order finite difference and C^0-linear Lagrange finite element techniques; however, if the second-order differential operator is replaced by two first-order operators to which piecewise linearization is applied, the resulting method may fail to converge owing to ill-conditioning and/or numerical errors. It has also been found that shooting techniques may fail to converge when solving two-point, linear, singular perturbation problems when the boundary layer thickness is very small.

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